

QUADRATIC FORMS



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INTRODUCTION

A **quadratic form** in two variables has the form:

$$ax^2 + by^2 + 2cxy,$$

where a, b, c come from some field. The most interesting field, and the one that ties this topic in with number theory, is the field of rational numbers, \mathbb{Q} .

For example $x^2 - y^2$ and xy are quadratic forms. Although they look quite different, the fact that:

$$x^2 - y^2 = (x + y)(x - y) = \mathbf{X}\mathbf{Y}$$

where $\mathbf{X} = x + y$ and $\mathbf{Y} = x - y$ means that they can be regarded as being equivalent.

A fundamental tool in studying quadratic forms is the matrix of a quadratic form. If we write $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $ax^2 + by^2 + 2cxy$ can be written as $\mathbf{v}^T \mathbf{A} \mathbf{v}$ where $\mathbf{A} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is a 2×2 symmetric matrix.

If $\mathbf{w} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \mathbf{P}\mathbf{v}$, where $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ then

$$\mathbf{v}^T \mathbf{A} \mathbf{v} = \mathbf{w}^T \mathbf{Q} \mathbf{w} \text{ where } \mathbf{Q} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}.$$

A fundamental problem is to classify quadratic forms up to equivalence. This is trivial over the fields \mathbb{C} , where there are only 3 equivalence classes of binary quadratic forms, with representatives 0 , x^2 and $x^2 + y^2$. That is, every binary quadratic form is equivalent to exactly one of these.

Over \mathbb{R} it is almost as trivial, where, in addition to these three, we have $-x^2$, $x^2 - y^2$ and $-x^2 - y^2$. But, over \mathbb{Q} , the problem is far from trivial.

CONTENTS

1. QUADRATIC FORMS

1.1 Quadratic Forms	7
1.2 Quadratic Spaces	8
1.3 Diagonalization of Quadratic Forms	10
1.4 Classification of Quadratic Forms	11
1.5 Determinant of a Quadratic Form	13

2. WITT'S DECOMPOSITION THEOREM

2.1 Direct Sums	15
2.2 Hyperbolic Spaces	16
2.3 Isotropic and Regular Spaces	17
2.4 Witt's Decomposition Theorem	20

3. WITT'S CANCELLATION THEOREM

3.1 Reflections	21
3.2 The Cancellation Theorem	22
3.3 The Witt Ring	23

4. QUADRATIC ALGEBRAS

4.1 Hamilton and His Quaternions	27
4.2 Quadratic Algebras	28
4.3 Quadratic Algebras and Quadratic Forms	29
4.4 The Witt Ring of a Finite Field	33

5. PFISTER FORMS

5.1 Pfister Forms	37
5.2 Isotropic Pfister Forms	39
5.3 The Characteristic of a Witt Ring	40
5.4 The Level of a Field	42