QUADRATIC FORMS

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INTRODUCTION

A **quadratic form** in two variables has the form:

$$ax^2 + by^2 + 2cxy,$$

where a, b, c come from some field. The most interesting field, and the one that ties this topic in with number theory, is the field of rational numbers, $\mathbb{Q}$.

For example $x^2 - y^2$ and xy are quadratic forms. Although they look quite different, the fact that:

$$x^2 - y^2 = (x + y)(x - y) = XY$$

where $X = x + y$ and $Y = x - y$ means that they can be regarded as being equivalent.

A fundamental tool in studying quadratic forms is the matrixof a quadratic form. If we write $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $ax^2 + by^2 + 2cxy$ can be written as $\mathbf{v}^T A\mathbf{v}$ where $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is a 2×2 symmetric matrix.

If $\mathbf{w} = \begin{pmatrix} X \\ Y \end{pmatrix} = P\mathbf{v}$, where $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ then $\mathbf{v}^T A\mathbf{v} = \mathbf{w}^T Q\mathbf{w}$ where $Q = \begin{pmatrix} 0 & ½ \\ ½ & 0 \end{pmatrix}$.

A fundamental problem is to classify quadratic forms up to equivalence. This is trivial over the fields $\mathbb{C}$, where there are only 3 equivalence classes of binary quadratic forms, with representatives 0, x^2 and $x^2 + y^2$. That is, every binary quadratic form is equivalent to exactly one of these.

Over $\mathbb{R}$ it is almost as trivial, where, in addition to these three, we have $-x^2$, $x^2 - y^2$ and $-x^2 - y^2$. But, over $\mathbb{Q}$, the problem is far from trivial.

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